EXISTENCE OF SOLUTIONS FOR A BOUNDARY VALUE PROBLEM ON AN INFINITE INTERVAL

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ABSTRACT. Based on a fixed point theorem due to Avery and Henderson, we prove that a second order boundary value problem has at least two positive solutions.

1. INTRODUCTION

Some of the most widely used theorems guaranteeing the existence of one or multiple fixed points are the ones due to Krasnoselskii [19], Leggett and Williams [10], and Avery and Henderson [3]. Among the latest additions to this series of theorems are the ones due to Avery, Henderson and O'Regan [4, 5, 6]. An innovating attempt to unify all the results mentioned above, carried out by Kwong, can be found in [9]. Roughly speaking, the essence of all these theorems is to generalize the Intermediate Value Theorem for real functions of one real variable to function spaces, which are Banach spaces of infinite dimensions. One very important aspect of this generalization is to properly transfer the meaning of the closed interval of the real line to such spaces. An excellent discussion on this subject can be found in [2, 9, 19].

This paper is a sequel of [15]. The main result presented in [15] is based on the Krasnoselskii Fixed Point Theorem and provides conditions which guarantee the existence of at least one nonnegative solution for the boundary value problem studied therein. Our goal in this paper is to achieve multiple solutions for the ordinary version of the same boundary value problem. To do this, we use a fixed point theorem due to Avery and Henderson. This theorem, apart from guarantying the existence of two fixed points, provides some additional information about them, which varies depending on the way it is used. Here, we obtain upper or lower boundaries for the values of these fixed points at two predefined points of their domain.

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Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}^+ := [0, +\infty)$. Also, for any interval $I \subseteq \mathbb{R}$ and any set $S \subseteq \mathbb{R}$, by $C(I, S)$ we denote the set of all continuous functions defined on I, which have values in S. Consider the second order nonlinear differential equation

$$x''(t) + f(t, x(t)) = 0, \quad t \in \mathbb{R}^+ \tag{1.1}$$

along with the initial condition

$$x(0) = 0 \tag{1.2}$$

and the boundary condition

$$\lim_{t \to +\infty} x'(t) = \xi \tag{1.3}$$

where f is a real valued function defined on the set $\mathbb{R}^+ \times \mathbb{R}$, which is increasing with respect to its second variable, nonnegative and continuous, and ξ is a nonnegative real number.

2. Preliminaries and Lemmas

Definition 2.1. A function $x \in C(\mathbb{R}^+, \mathbb{R})$ is a solution of the boundary value problem (1.1) – (1.3) if x is twice continuously differentiable and satisfies equation (1.1) and the boundary condition (1.3).

Definition 2.2. Let E be a real Banach space. A cone in E is a nonempty, closed set $P \subseteq E$ such that

(*i*) $\kappa u + \lambda v \in P$ for all $u, v \in P$ and all $\kappa, \lambda \geq 0$,
(*ii*) $u, -u \in P$ implies $u = 0$.

Definition 2.3. Let P be a cone in a real Banach space E. A functional $\psi : P \to E$ is said to be increasing on P if $\psi(x) \leq \psi(y)$, for any $x, y \in P$ with $x \leq y$, where $\leq$ is the partial ordering induced to the Banach space by the cone P, i.e.

$$x \leq y \quad \text{if and only if} \quad y - x \in P.$$

Definition 2.4. Let ψ be a nonnegative functional on a cone P. For each $d > 0$, we denote by $P(\psi, d)$ the set

$$P(\psi, d) := \{x \in P : \psi(x) < d\}.$$

The results of this paper are based on the following fixed point theorem, due to Avery and Henderson [3].

Theorem 2.5. *Let P be a cone in a real Banach space E. Let α and γ be increasing, nonnegative, continuous functionals on P, and let θ be a nonnegative functional on P with $\theta(0) = 0$ such that, for some $c > 0$ and $\Theta > 0$,*

$$\gamma(x) \leq \theta(x) \leq \alpha(x) \quad \text{and} \quad \|x\| \leq \Theta\gamma(x),$$

for all $x \in \overline{P(\gamma, c)}$. *Suppose there exists a completely continuous operator* $A : \overline{P(\gamma, c)} \to P$ *and real constants* a, b *with* $0 < a < b < c$, *such that*

$$\theta(\lambda x) \leq \lambda \theta(x), \quad \textit{for} \quad 0 \leq \lambda \leq 1 \quad \textit{and} \quad x \in \partial P(\theta, b),$$

and either

(*i*) $\gamma(Ax) > c$, *for all* $x \in \partial P(\gamma, c)$,
(*ii*) $\theta(Ax) < b$, *for all* $x \in \partial P(\theta, b)$,
(*iii*) $P(\alpha, a) \neq \emptyset$, *and* $\alpha(Ax) > a$, *for all* $x \in \partial P(\alpha, a)$

or

(*i*) $\gamma(Ax) < c$, *for all* $x \in \partial P(\gamma, c)$,
(*ii*) $\theta(Ax) > b$, *for all* $x \in \partial P(\theta, b)$,
(*iii*) $P(\alpha, a) \neq \emptyset$, *and* $\alpha(Ax) < a$, *for all* $x \in \partial P(\alpha, a)$.

Then A *has at least two fixed points* x_1 *and* x_2 *belonging to* $\overline{P(\gamma, c)}$ *such that*

$$a < \alpha(x_1), \quad \textit{with} \quad \theta(x_1) < b,$$

and

$$b < \theta(x_2), \quad \textit{with} \quad \gamma(x_2) < c.$$

3. Main Results

Let $BC(\mathbb{R}^+, \mathbb{R})$ be the Banach space of all bounded continuous real valued functions on the interval $\mathbb{R}^+$, endowed with the sup-norm $\|\cdot\|$ defined by

$$\|u\| := \sup_{t \geq 0} |u(t)|, \quad \text{for} \quad u \in BC(\mathbb{R}^+, \mathbb{R}).$$

Definition 3.1. A set U of real valued functions defined on the interval $\mathbb{R}^+$ is called equiconvergent at ∞ if all functions in U are convergent in $\mathbb{R}$ at the point ∞ and, in addition, for each $\epsilon > 0$, there exists $T \equiv T(\epsilon) > 0$ such that, for all functions $u \in U$, it holds

$$|u(t) - \lim_{s \to \infty} u(s)| < \epsilon, \quad \text{for every} \quad t \geq T.$$

Lemma 3.2. *Let* U *be an equicontinuous and uniformly bounded subset of the Banach space* $BC(\mathbb{R}^+, \mathbb{R})$. *If* U *is equiconvergent at* ∞, *it is also relatively compact.*

Let

$$E = \{y \in C(\mathbb{R}^+, \mathbb{R}) : \ y(t) = O(t) \ \ \text{for} \ \ t \to +\infty\}.$$

The set E is a real Banach space endowed with the norm $\|\cdot\|_E$, defined by

$$\|y\|_E := \sup_{t\geq 0}\frac{|y(t)|}{t+1}, \quad \text{for every} \quad y \in E.$$

Also, we define the following set K, which is a cone in E

$$K := \{x \in E : \ x(0) = 0, \ x(t) \geq \min\{t, 1\}\|x\|_E, \ \text{for } t \in \mathbb{R}^+, \\ \text{and } x \text{ is nondecreasing}\}.$$

Let

$$0 < r_1 \leq r_2 \leq r_3 \leq 1$$

and consider the following functionals

$$\gamma(x) = x(r_1), \quad x \in K$$
$$\theta(x) = x(r_2), \quad x \in K$$

and

$$\alpha(x) = x(r_3), \quad x \in K.$$

It is easy to see that α, γ are nonnegative, increasing and continuous functionals on K, θ is nonnegative on K and $\theta(0) = 0$. Also, it is straightforward that

$$\gamma(x) \leq \theta(x) \leq \alpha(x),$$

since $x \in K$ is nondecreasing on $\mathbb{R}^+$. Furthermore, for any $x \in K$, we have

$$\gamma(x) = x(r_1) \geq r_1\|x\|_E,$$

so

$$\|x\|_E \leq \frac{1}{r_1}\gamma(x), \quad x \in K.$$

Additionally, by the definition of θ it is obvious that

$$\theta(\lambda x) = \lambda\theta(x), \quad 0 \leq \lambda \leq 1, \quad x \in K.$$

At this point, we state the following assumptions.

($\mathbf{H_1}$) There exists $M > \xi$, a continuous function $u : \mathbb{R}^+ \to \mathbb{R}^+$ and a nondecreasing function $L : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(t, y) \leq u(t)L\left(\frac{y}{1+t}\right), \quad t \in \mathbb{R}^+, \quad y \in \mathbb{R}^+$$

and also

$$\xi r_2 + L(M)\left[\int_0^{r_2} su(s)ds + r_2\int_{r_2}^{\infty} u(s)ds\right] < Mr_2.$$

(H_2) There exist a constant $\delta \in (0,1]$, a continuous function $v : \mathbb{R}^+ \to \mathbb{R}^+$ and a nondecreasing function $w : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(t,y) \geq v(t)w(y), \quad t \in [\delta, +\infty), \quad y \in \mathbb{R}^+.$$

(H_3) There exist $\rho_1, \rho_3 > 0$ such that

$$\frac{\rho_i}{\delta}(r_i+1) < \xi r_i + w(\rho_i)\left[\int_{[0,r_i]\cap[\delta,+\infty)} sv(s)ds + r_i \int_{[r_i,\infty)\cap[\delta,+\infty)} v(s)ds\right],$$

for $i = 1, 3$ and

$$\frac{\rho_3}{\delta}(r_3+1) < Mr_2 < \frac{\rho_1}{\delta}(r_1+1).$$

Lemma 3.3. *Suppose that assumption (H_1) holds and let $\epsilon > 0$. A function $x \in \overline{K(\gamma,\epsilon)}$ is a solution of the boundary value problem* (1.1)–(1.3) *if and only if x is a fixed point of the operator $A : \overline{K(\gamma,\epsilon)} \to C(\mathbb{R}^+, \mathbb{R})$, defined by the formula*

$$\text{(3.1)} \quad Ay(t) := \xi t + \int_0^\infty \min\{t,s\} f(s,y(s))ds, \quad \textit{for every } t \in \mathbb{R}^+,$$

or, equivalently,

(3.2)

$$Ay(t) := \xi t + \int_0^t sf(s,y(s))ds + t\int_t^\infty f(s,y(s))ds, \quad \textit{for every } t \in \mathbb{R}^+.$$

Proof. First of all, we will show that operator A is well defined. Indeed, for any $\epsilon > 0$ and any $x \in \overline{K(\gamma,\epsilon)}$ we have

$$r_1\|x\|_E \leq x(r_1) \leq \epsilon$$

and

$$\|x\|_E \leq \frac{\epsilon}{r_1}.$$

Also, for every $t \in \mathbb{R}^+$, it holds that

$$\frac{x(t)}{1+t} \leq \sup_{\sigma\in\mathbb{R}^+} \frac{x(\sigma)}{1+\sigma} = \|x\|_E \leq \frac{\epsilon}{r_1}.$$

Consequently, for any $t \in \mathbb{R}^+$, using assumption (H_1), we have

$$f(t,x(t)) \leq u(t)L\left(\frac{x(t)}{1+t}\right) \leq u(t)L\left(\frac{\epsilon}{r_1}\right),$$

therefore,

$$\int_0^\infty f(s,x(s))ds \leq \int_0^\infty u(s)L\left(\frac{\epsilon}{r_1}\right)ds = L\left(\frac{\epsilon}{r_1}\right)\int_0^\infty u(s)ds < \infty.$$

Hence, the formula of operator A makes sense for any $x \in \overline{K(\gamma,\epsilon)}$.

For the rest of the proof, see [16]. □

Lemma 3.4. *Suppose that assumption (H_1) holds. Then, the operator A is completely continuous and, for every $\epsilon > 0$, maps $\overline{K(\gamma,\epsilon)}$ into K.*

Proof. First, we will show that A maps $\overline{K(\gamma,\epsilon)}$ into K. Let $x \in \overline{K(\gamma,\epsilon)}$. Then obviously $Ax(t) \geq 0$ for every $t \in \mathbb{R}^+$, and $Ax(0) = 0$. Additionally,

$$(Ax)'(t) = \xi + \int_t^\infty f(s,x(s))ds \geq 0, \quad \text{for every} \quad t \in \mathbb{R}^+.$$

Next, we observe that, for any nonnegative real numbers t and σ, it holds

$$t \geq \begin{cases} \frac{t}{\sigma+1}\sigma & \text{for } t \in [0,1], \\ \frac{1}{\sigma+1}\sigma & \text{for } t \in [1,\infty). \end{cases}$$

That is

$$t \geq \frac{\min\{t,1\}}{\sigma+1}\sigma, \quad \text{for every } t \geq 0 \text{ and } \sigma \geq 0. \tag{3.3}$$

Moreover, it is not difficult to verify that, if t, s, σ are arbitrary nonnegative real numbers, then

$$\min\{t,s\} \geq \begin{cases} \frac{t}{\sigma+1}\min\{\sigma,s\} & \text{for } t \in [0,1], \\ \frac{1}{\sigma+1}\min\{\sigma,s\} & \text{for } t \in [1,\infty). \end{cases}$$

Namely, we have

$$\min\{t,s\} \geq \frac{\min\{t,1\}}{\sigma+1}\min\{s,\sigma\}, \quad \text{for every } t,s,\sigma \geq 0. \tag{3.4}$$

Since the function f is nonnegative and using (3.3) and (3.4), we obtain, for every $t \geq 0$ and $\sigma \geq 0$,

$$\begin{aligned} Ax(t) =& \xi t + \int_0^\infty \min\{t,s\} f(s,x(s))ds \\ \geq& \xi\frac{\min\{t,1\}}{\sigma+1}\sigma + \frac{\min\{t,1\}}{\sigma+1}\int_0^\infty \min\{\sigma,s\}f(s,x(s))ds \\ =& \min\{t,1\}\left\{\frac{1}{\sigma+1}\left(\xi\sigma + \int_0^\infty \min\{\sigma,s\}f(s,x(s))ds\right)\right\} \\ =& \min\{t,1\}\frac{Ax(\sigma)}{\sigma+1}. \end{aligned}$$

Therefore,

$$Ax(t) \geq \min\{t,1\}\sup_{\sigma\geq 0}\frac{Ax(\sigma)}{\sigma+1}, \quad \text{for every } t \geq 0,$$

i.e.

$$Ax(t) \geq \min\{t,1\}\|Ax\|_E, \quad \text{for every } t \geq 0.$$

Consequently $Ax \in K$.

Also, similarly to [16], we can prove that $A(\overline{K(\gamma, c)})$ is relatively compact and A is continuous. So, we have proved that the operator A is completely continuous. □

Theorem 3.5. *Suppose that assumptions (H_1)-(H_3) hold. Then the boundary value problem* (1.1)-(1.3) *has at least two nondecreasing, concave and positive on* $\mathbb{R}^+$ *solutions* $x, \tilde{x}$ *such that*

$$x(r_3) > \frac{\rho_3}{\delta}(r_3 + 1), \quad x(r_2) < Mr_2$$

and

$$\tilde{x}(r_1) < \frac{\rho_1}{\delta}(r_1 + 1), \quad \tilde{x}(r_2) > Mr_2.$$

Proof. Set $a = \frac{\rho_3}{\delta}(r_3 + 1)$, $b = Mr_2$ and $c = \frac{\rho_1}{\delta}(r_1 + 1)$. From Lemma 3.4, we have that A is a completely continuous operator, which maps $\overline{K(\gamma, c)}$ into K.

Now, let $x \in \partial K(\gamma, c)$. Then $\gamma(x) = x(r_1) = c$, so

$$\|x\|_E \geq \frac{c}{r_1 + 1}. \tag{3.5}$$

Having in mind assumption ($\mathrm{H_2}$), we get

$$\begin{aligned}
\gamma(Ax) &= Ax(r_1) \\
&= \xi r_1 + \int_0^{r_1} sf(s, x(s))ds + r_1 \int_{r_1}^{\infty} f(s, x(s))ds \\
&\geq \xi r_1 + \int_{[0,r_1]\cap[\delta,+\infty)} sf(s, x(s))ds + r_1 \int_{[r_1,\infty)\cap[\delta,+\infty)} f(s, x(s))ds \\
&\geq \xi r_1 + \int_{[0,r_1]\cap[\delta,+\infty)} sv(s)w(x(s))ds + r_1 \int_{[r_1,\infty)\cap[\delta,+\infty)} v(s)w(x(s))ds \\
&\geq \xi r_1 + \int_{[0,r_1]\cap[\delta,+\infty)} sv(s)w(x(\delta))ds + r_1 \int_{[r_1,\infty)\cap[\delta,+\infty)} v(s)w(x(\delta))ds.
\end{aligned}$$

So, since $x \in K$, we have

$$\begin{aligned}
\gamma(Ax) &\geq \xi r_1 + \int_{[0,r_1]\cap[\delta,+\infty)} sv(s)w(\delta\|x\|_E)ds + r_1 \int_{[r_1,\infty)\cap[\delta,+\infty)} v(s)w(\delta\|x\|_E)ds \\
&= \xi r_1 + w(\delta\|x\|_E)\left[\int_{[0,r_1]\cap[\delta,+\infty)} sv(s)ds + r_1 \int_{[r_1,\infty)\cap[\delta,+\infty)} v(s)ds\right].
\end{aligned}$$

At this point, we use (3.5) and we get

$$\begin{aligned}\gamma(Ax) &\geq \xi r_1 + w(\delta\frac{c}{r_1+1})\left[\int_{[0,r_1]\cap[\delta,+\infty)} sv(s)ds + r_1\int_{[r_1,\infty)\cap[\delta,+\infty)} v(s)ds\right]\\ &= \xi r_1 + w(\rho_1)\left[\int_{[0,r_1]\cap[\delta,+\infty)} sv(s)ds + r_1\int_{[r_1,\infty)\cap[\delta,+\infty)} v(s)ds\right].\end{aligned}$$

Using hypothesis ($\mathbf{H_3}$), we conclude that

$$\gamma(Ax) > \frac{\rho_1}{\delta}(r_1+1),$$

so condition (i) of Theorem 2.5 is satisfied.

Now let $x \in \partial K(\theta, b)$. Then $\theta(x) = x(r_2) = b$, so since $x \in K$, we have

$$\|x\|_E \leq \frac{x(r_2)}{r_2} = \frac{b}{r_2} = M.$$

Consequently, by assumption ($\mathbf{H_1}$), we have

$$\begin{aligned}\theta(Ax) &= Ax(r_2)\\ &= \xi r_2 + \int_0^{r_2} sf(s,x(s))ds + r_2\int_{r_2}^{\infty} f(s,x(s))ds\\ &\leq \xi r_2 + \int_0^{r_2} su(s)L\left(\frac{x(s)}{1+s}\right)ds + r_2\int_{r_2}^{\infty} u(s)L\left(\frac{x(s)}{1+s}\right)ds\\ &\leq \xi r_2 + \int_0^{r_2} su(s)L(M)ds + r_2\int_{r_2}^{\infty} u(s)L(M)ds\\ &= \xi r_2 + L(M)\left[\int_0^{r_2} su(s)ds + r_2\int_{r_2}^{\infty} u(s)ds\right].\end{aligned}$$

So

$$\theta(Ax) \leq Mr_2 = b,$$

which means that condition (ii) of Theorem 2.5 is satisfied.

Now, define the function $y : \mathbb{R}^+ \to \mathbb{R}$ with $y(t) = \frac{a}{2}$. Then, it is obvious that $\alpha(y) = \frac{a}{2} < a$, so $K(\alpha, a) \neq \emptyset$. Also, since $\alpha(x) = x(r_3) = a$, we have $\frac{x(r_3)}{r_3+1} = \frac{a}{r_3+1}$, so

$$\|x\|_E \geq \frac{a}{r_3+1}. \tag{3.6}$$

As in the case of the functional γ above, we get

$$\begin{aligned}\alpha(Ax) &= Ax(r_3)\\ &\geq \xi r_3 + \int_{[0,r_3]\cap[\delta,+\infty)} sv(s)w(x(\delta))ds + r_3\int_{[r_3,\infty)\cap[\delta,+\infty)} v(s)w(x(\delta))ds.\end{aligned}$$

So, since $x \in K$, we have

$$\alpha(Ax) \geq \xi r_3 + w(\delta\|x\|_E)\left[\int_{[0,r_3]\cap[\delta,+\infty)} sv(s)ds + r_3\int_{[r_3,\infty)\cap[\delta,+\infty)} v(s)ds\right]$$

and using (3.6) we get

$$\alpha(Ax) \geq \xi r_3 + w(\rho_3)\left[\int_{[0,r_3]\cap[\delta,+\infty)} sv(s)ds + r_3\int_{[r_3,\infty)\cap[\delta,+\infty)} v(s)ds\right].$$

Therefore, by hypothesis ($\mathrm{H_3}$), we conclude that

$$\alpha(Ax) > \frac{\rho_3}{\delta}(r_3+1),$$

so condition *(iii)* of Theorem 2.5 is satisfied.

At this point, we apply Theorem 2.5 to obtain that operator A has at least two fixed points x and $\tilde{x}$ belonging to $\overline{K(\gamma, c)}$ such that

$$x(r_3) > \frac{\rho_3}{\delta}(r_3+1), \quad x(r_2) < Mr_2$$

and

$$\tilde{x}(r_1) < \frac{\rho_1}{\delta}(r_1+1), \quad \tilde{x}(r_2) > Mr_2.$$

This concludes the proof. □

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